Moments of Traces for Circular β-ensembles

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This is joint work with Sho Matsumoto

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Outline

- Moments for Haar Unitary Matrices (D.E. Thm)
- Background for Circular β-Ensembles
- Moments for Circular β-Ensembles
- Proofs by Jack Polynomials

1. Moments for Haar Unitary Matrices

- What is Haar-invariant unitary matrix Γ_n?
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  1) The matrix Q in QR (Gram-Schmidt) decomposition of Y

  2) $\Gamma_n \stackrel{d}{=} Y(Y^*Y)^{-1/2}$

▶ Theorem (Diaconis and Evans: 2001)

(a) $a = (a_1, \quad , a_k)$, $b = (b_1, \quad , b_k)$ with $a_j, b_j \in \{0, 1, 2, \quad \}$.
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$X_1, \quad , X_k$: i.i.d. $\mathbb{C}N(0,1)$. If $n \quad \sum_{j=1}^{k} ja_j \_ \sum_{j=1}^{k} jb_j$,

$$\mathbb{E}\Big[\prod_{j=1}^{k}(Tr(U_n^j))^{a_j}\overline{(Tr(U_n^j))^{b_j}}\Big]$$

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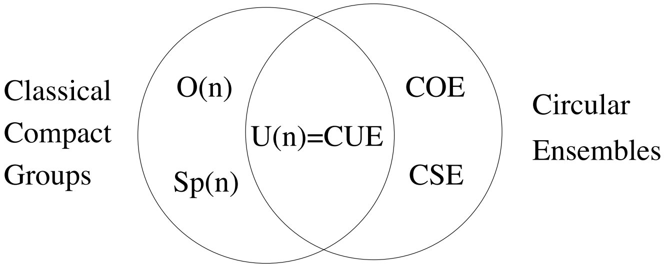
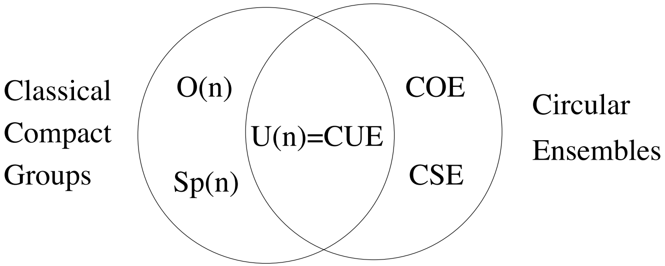
(b) For j and k,

$$
\mathbb{E}\big[Tr(U_n^j)\overline{Tr(U_n^k)}\big] = \delta_{jk} \quad j \wedge n.
$$



Circular Ensembles and Haar-invariant Matrices from Classical Compact Groups



Circular Ensembles and Haar-invariant Matrices from Classical Compact Groups

Diaconis (2004) believes there is a good formula for COE and CSE

2. Background for Circular β-Ensembles

- Probability density function

  $e^{i\theta_1}, \quad , e^{i\theta_n}$: eigenvalues of Haar-invariant unitary matrix.

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$$f(\theta_1, \quad , \theta_n j \beta) = Const \prod_{1 \le j < k \le n} j e^{i\theta_j} \quad e^{i\theta_k} j^{\beta}$$

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  - This model: *circular β-ensemble* ($\beta = 1, 2, 4$) by physicist Dyson for study of nuclear scattering data

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Killip & Nenciu: Matrix models for circular ensembles

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  - This suggest: moments for general β-ensemble depend on n

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$\lambda = (3, 2, 2) : |\lambda| = 7, m_2(\lambda) = 2,\ m_3(\lambda) = 1, l(\lambda) = 3,$

$$p_\lambda = (\textstyle\sum_i \lambda_i^3) \quad (\textstyle\sum_i \lambda_i^2)^2$$

$\alpha > 0$, $K \quad 1$, $n \quad 1$, define

$$A = \left(1 \quad \frac{j\alpha \quad 1j}{n \quad K + \alpha}\,\delta(\alpha \quad 1)\right)^K$$

$$B = \left(1 + \frac{j\alpha \quad 1j}{n \quad K + \alpha}\,\delta(\alpha < 1)\right)^K$$

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Let $\theta_1, \quad , \theta_n \quad f(\theta_1, \quad , \theta_n j\beta)$, $\alpha = 2/\beta$.

- $Z_n = (e^{i\theta_1}, \quad , e^{i\theta_n})$,
- $p_\mu(Z_n) = p_\mu(e^{i\theta_1}, \quad , e^{i\theta_n})$

Theorem

(a) *If* $n \quad K = j\mu j$*, then*

$$A \quad \frac{\mathbb{E}\left[j p_\mu(Z_n) j^2\right]}{\alpha^{l(\mu)} z_\mu} \quad B$$

Theorem

(a) *If* $n \quad K = |\mu|$, *then*

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If μ 6$=$ ν *and* $n \quad K = j\mu j$ _ $j\nu j$, *then*

$$\left| \mathbb{E}\left[p_\mu(Z_n)\overline{p_\nu(Z_n)} \right] \right| \quad \max f jA \quad 1j, jB \quad 1jg\, \alpha^{(l(\mu)+l(\nu))/2} (z_\mu z_\nu)^{1/2}$$

Theorem

(a) *If* $n \quad K = j\mu j$, *then*

$$A \quad \frac{\mathbb{E}\left[jp_\mu(Z_n)j^2 \right]}{\alpha^{l(\mu)} z_\mu} \quad B$$

(b) *If* $j\mu j \; 6\!= j\nu j$, *then* $\mathbb{E}\left[p_\mu(Z_n)\overline{p_\nu(Z_n)} \right] = 0$.
If $\mu \; 6\!= \nu$ *and* $n \quad K = j\mu j \,\_\, j\nu j$, *then*

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(c) $9\, C = C(\beta)$ *s.t.* $8m \quad 1, \; n \quad 2$

$$\left| \mathbb{E}\left[jp_m(Z_n)j^2 \right] \quad n \right| \quad C\frac{n^3 2^{n\beta}}{m^{1\wedge\beta}}$$

Take $\beta = 2$, then $A = B = 1$. We recover

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  For $n \quad \sum_{j=1}^{k} ja_j$ _ $\sum_{j=1}^{k} jb_j$,

$$\mathbb{E}\left[\prod_{j=1}^{k}(Tr(U_n^j))^{a_j}\overline{(Tr(U_n^j))^{b_j}}\right] = \delta_{ab}\prod_{j=1}^{k} j^{a_j} a_j!$$

Corollary

8 $\beta > 0$,

$$(a) \quad \lim_{n\to\infty} \mathbb{E}\left[p_\mu(Z_n)\overline{p_\nu(Z_n)}\right] = \delta_{\mu\nu}\left(\frac{2}{\beta}\right)^{l(\mu)} z_\mu;$$

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$$(b) \quad \lim_{m\to\infty} \mathbb{E}\left[jp_m(Z_n)j^2\right] = n \quad \textit{for any } n \quad 2.$$

Corollary

$\mu \neq \nu$: $K = |\mu| \vee |\nu|$. If $n \geq 2K$, then

$$(a) \quad \left| \frac{\mathbb{E}[|p_\mu(Z_n)|^2]}{\alpha^{l(\mu)} z_\mu} - 1 \right| \leq \frac{6|1-\alpha|K}{n};$$

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  The exact formula gives

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$$\mathbb{E}[|p_1(Z_n)|^2] = \frac{2}{\beta}\frac{n}{n - 1 + 2\beta^{-1}} = \begin{cases} \frac{2n}{n+1}, & \text{if } \beta = 1 \\ 1, & \text{if } \beta = 2 \\ \frac{n}{2n-1}, & \text{if } \beta = 4 \end{cases}$$

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Exact formula is given next

Proofs by Jack Polynomial

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  Jack polynomial $J_\lambda^{(\alpha)} = J_\lambda^{(\alpha)}(x_1, \quad , x_n)$ is symmetric in $x_1, \quad , x_n$

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  Orthogonal property: $Z_n = (e^{i\theta_1}, \ldots, e^{i\theta_n})$

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  Jack polynomial $J_{\lambda}^{(\alpha)} = J_{\lambda}^{(\alpha)}(x_1$

Write

$$J_\lambda^{(\alpha)} = \sum_{\rho:|\rho|=|\lambda|} \theta_\rho^\lambda(\alpha) p_\rho$$

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For $|\mu| = |\nu| = K$,

$$\mathbb{E}\left[p_\mu(Z_n) \overline{p_\nu(Z_n)} \right] = \sum_{\lambda \vdash K :\, l(\lambda) \le n} \Theta_\mu^\lambda(\alpha) \Theta_\nu^\lambda(\alpha) \mathbb{E}(J_\lambda^{(\alpha)} \overline{J_\lambda^{(\alpha)}})$$

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= \quad &\alpha^{l(\mu)+l(\nu)} z_\mu z_\nu \sum_{\lambda\vdash K:\, l(\lambda)\le n} \frac{\theta_\mu^\lambda(\alpha)\theta_\nu^\lambda(\alpha)}{C_\lambda(\alpha)} \mathsf{N}_\lambda^\alpha(n)
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$$C_\lambda(\alpha) = \prod_{(i,j)\in\lambda} \Big\{(\alpha(\lambda_i \quad j) + \lambda_j' \quad i + 1)(\alpha(\lambda_i \quad j) + \lambda_j' \quad i + \alpha)\Big\}$$

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$$N_\lambda^\alpha(n) = \prod_{(i,j)\in\lambda} \frac{n + (j \quad 1)\alpha \quad (i \quad 1)}{n + j\alpha \quad i}$$

Young diagram

Main proof:

- play $C_\lambda(\alpha)$
- play $N_\lambda^\alpha(n)$
- use orthogonal relations of $\theta_\mu^\lambda(\alpha)$

- Examples

$$\mathbb{E}[|p_1(Z_n)|^4] = \frac{2n\alpha^2(n^2 + 2(\alpha - 1)n - \alpha)}{(n+\alpha-1)(n+\alpha-2)(n+2\alpha-1)}$$

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$$
\begin{aligned}
\mathbb{E}[j p_1(Z_n) j^4] &= \frac{2n\alpha^2(n^2 + 2(\alpha \quad 1)n \quad \alpha)}{(n+\alpha \quad 1)(n+\alpha \quad 2)(n+2\alpha \quad 1)} \\
&= \begin{cases} \frac{8(n^2+2n-2)}{(n+1)(n+3)}, & \text{if } \beta = 1 \\ 2, & \text{if } \beta = 2 \\ \frac{2n^2-2n-1}{(2n-1)(2n-3)}, & \text{if } \beta = 4 \end{cases}
\end{aligned}
$$

$$\mathbb{E}\left[p_2(Z_n)\overline{p_1(Z_n)^2}\right]$$

$$
\begin{aligned}
& \mathbb{E}\left[p_2(Z_n)\overline{p_1(Z_n)^2} \right] \\
= \quad & \frac{2\alpha^2(\alpha \quad 1)n}{(n+\alpha \quad 1)(n+2\alpha \quad 1)(n+\alpha \quad 2)}
\end{aligned}
$$

$$
\begin{aligned}
& \mathbb{E}\left[p_2(Z_n)\overline{p_1(Z_n)^2}\right] \\
= \quad & \frac{2\alpha^2(\alpha \quad 1)n}{(n+\alpha \quad 1)(n+2\alpha \quad 1)(n+\alpha \quad 2)} \\
= \quad & \begin{cases} \frac{8}{(n+1)(n+3)}, & \text{if } \beta = 1 \\ 0, & \text{if } \beta = 2 \\ \frac{-1}{(2n-1)(2n-3)}, & \text{if } \beta = 4 \end{cases}
\end{aligned}
$$

The End!

Thanks for your patience!